Quotient interleaving distances

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Jan. 31, 2020

Outline:

- A categorical framework for interleaving distances.
- A general stability result.
- Prove metric properties: completeness, intrinsicness, etc.

Locally persistent categories

Let $I = ([0, \infty), \leq, +)$ monoidal poset.

Definition: The category of **persistent sets** is $\mathbf{Set}^I$.

Definition: A **locally persistent category** is a category enriched in persistent sets. Concretely, $\mathcal{C} \in \mathbf{lpCat}$ consists of:

- a collection of objects $\mathrm{obj}(\mathcal{C})$;
- for $x, y \in \mathrm{obj}(\mathcal{C})$, $r \in I$, a set of **morphisms of length r**
$$\mathrm{Hom}_{\mathcal{C}}(x, y)_r;$$
- for $x \in \mathrm{obj}(\mathcal{C})$, an **identity morphism** $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)_0$;
- for $x, y, z \in \mathrm{obj}(\mathcal{C})$, $r, s \in I$, a **composition operation**
$$- \circ - : \mathrm{Hom}_{\mathcal{C}}(y, z)_s \times \mathrm{Hom}_{\mathcal{C}}(x, y)_r \to \mathrm{Hom}_{\mathcal{C}}(x, z)_{r+s};$$
- for $x, y \in \mathrm{obj}(\mathcal{C})$, $r \leq s \in I$, a **shift operation**
$$S_{r,s} : \mathrm{Hom}_{\mathcal{C}}(x, y)_r \to \mathrm{Hom}_{\mathcal{C}}(x, y)_s;$$

S.t. unitality, associativity, S is functorial, $\circ$ and S are compatible.

Examples of locally persistent categories

Example: The category **Met** of extended pseudo metric spaces.

$$\mathrm{Hom}(P,Q)_r := \{f : P \to Q \mid d_P(x,y) + r \geq d_Q(f(x), f(y))\}.$$

Example: Any category $C^{\mathbb{R}}$ for $C \in$ **Cat** and $\mathbb{R} = (\mathbb{R}, \leq)$.

$$\mathrm{Hom}(X,Y)_r := \mathrm{Nat}(X, Y[r]).$$

Example: More generally, any category with a flow $(C, \mathcal{T})$.

$$\mathrm{Hom}(x,y)_r := \mathrm{Hom}_{\mathcal{C}}(x, \mathcal{T}_r(y)).$$

Interleaving distance

Let $\mathcal{C} \in \mathbf{IpCat}$, $x, y \in \mathcal{C}$, $\delta \in I$.

Definition: An **δ-interleaving between x and y** is given by:

$$f \in \mathrm{Hom}_{\mathcal{C}}(x, y)_\delta,\ g \in \mathrm{Hom}_{\mathcal{C}}(y, x)_\delta,\ \text{ s.t.}$$
$$g \circ f = S_{0,2\delta}(\mathrm{id}_x),\ f \circ g = S_{0,2\delta}(\mathrm{id}_y).$$

Lemma: The following defines an extended pseudo metric on $\mathrm{obj}(\mathcal{C})$, called the **interleaving distance**.

$$d_I^{\mathcal{C}}(x, y) = \inf\{\delta \in I \mid x \text{ and } y \text{ are } \delta\text{-interleaved}\}$$

Remarks:

- In $C^{\mathbb{R}}$, d_I is the usual interleaving distance.
- In $\mathbf{Top}^{\mathbb{R}}$, d_I is not homotopy invariant.
- In **Met**, P, Q are δ-interleaved if and only if there is a bijection that doesn't distort the metric more than δ.

Quotient metrics

Let $R \subseteq X \times X$ equivalence relation, and $d : X \times X \to [0, \infty]$ an extended pseudo metric.

Definition: d is ***R*-invariant** if $xRx', yRy' \Rightarrow d(x, y) = d(x', y')$.

Definition: The **quotient metric** $d_{/R} : X \times X \to [0, \infty]$ is the largest R-invariant metric bounded above by d.

Definition: Given $P \in \mathbf{Met}$, let $\overline{P} \in \mathbf{Met}$ be given by identifying points at distance 0. Let $P \simeq Q$ if $\overline{P}$ and $\overline{Q}$ are isometric.

Theorem (S.): $\left(d_I^{\mathbf{Met}}\right)_{/\simeq}(P, Q) = 2d_{GH}(P, Q)$.

Definition: For $X, Y \in \mathbf{Top}^{\mathbb{R}}$, $X \simeq Y$ if they are weakly equivalent in the projective model structure.

Proposition: $\left(d_I^{\mathbf{Top}^{\mathbb{R}}}\right)_{/\simeq}(X, Y) = d_{HI}(X, Y)$, the Homotopy Interleaving distance of Blumberg-Lesnick.

Stability results

Let $\mathcal{C}, \mathcal{D} \in$ **IpCat** and R, S, equivalence relations on obj($\mathcal{C}$) and obj($\mathcal{D}$) respectively.

Lemma: $F : \mathcal{C} \to \mathcal{D}$, $\mathbf{Set}^I$-enriched, that maps R-related objects to S-related objects. Then F is 1-Lipschitz wrt $(d_I^{\mathcal{C}})_{/R}$ and $(d_I^{\mathcal{D}})_{/S}$.

Applications:

- Vietoris-Rips and Čech $\mathbf{Met} \to \mathbf{Top}^{\mathbb{R}}$ are 2-Lipschitz wrt Gromov-Hausdorff distance and Homotopy Interleaving distance.
- Persistent homology $\mathbf{Top}^{\mathbb{R}} \to \mathbf{Vec}^{\mathbb{R}}$ is 1-Lipschitz wrt Homotopy Interleaving distance and interleaving distance.
- Degree-Rips is stable.
- Some versions of Mapper are stable.
- Spatio-temporal Rips filtration is stable (due to Kim-Mémoli).
- Extend the above to dynamic metric measure spaces.

Metric properties

Theorem (S.): Let $\mathcal{C} \in$ **lpCat** and R an equivalence relation on obj($\mathcal{C}$). If $\mathcal{C}$ is complete or cocomplete, then $(d_I^{\mathcal{C}})_{/R}$ is intrinsic.

Let $C \in$ **Cat** and $\mathcal{W} \subseteq C$ a class of morphisms.
Say that $x \simeq y$ if x and y are connected by a zig-zag in $\mathcal{W}$.

Theorem (S.): Let $\mathcal{C} \in$ **lpCat** and $\mathcal{W}$ a class of morphisms of length 0. If $\mathcal{C}$ has weighted sequential limits and $\mathcal{W}$ is closed under weighted pullbacks and sequential limits, then $(d_I^{\mathcal{C}})_{/\simeq}$ is complete.

In $\mathbf{Top}^{\mathbb{R}}$, let $\mathcal{W}$ be the class of projective trivial fibrations.
In **Met**, let $\mathcal{W}$ be the class of surjective distance-preserving maps.

Corollary (S.): The Homotopy Interleaving distance is intrinsic and complete.
Corollary: The Gromov-Hausdorff distance is intrinsic and complete.

Further results

Currently working on similar results to deduce:

- geodesicness;
- non-pseudoness (i.e. $d(x, y) = 0$ implies $x \simeq y$).

Thank you for your attention!