The Yoneda embedding for quasicategories

Luis Scoccola

Reading seminar on Higher Category Theory
University of Western Ontario, Fall 2016

The goal of this talk is to introduce two different ways in which the Yoneda embedding can be constructed in the higher categorical setting. Our references are:

- [Lur09] for the construction involving simplicially enriched categories.
- [Joy08] and [Joy09] for the construction using the universal left fibration.

To be able to define the Yoneda embedding we first need to find a suitable replacement for the category of sets and for the notion of a presheaf. The former is given by the *quasicategory of spaces* $\mathcal{S}$, defined as the homotopy coherent nerve $\mathbf{N}(\mathsf{Kan})$ of the simplicial category Kan of Kan complexes. Consequently, by a *presheaf* on a quasicategory C we will understand a simplicial map $C^{op} \to \mathcal{S}$, and the quasicategory of presheaves is the exponential $\mathcal{P}(C) := \mathcal{S}^{C^{op}}$. Since $\mathcal{S}$ is a quasicategory this is again a quasicategory.

We want to construct a simplicial map $C \to \mathcal{P}(C)$. By definition of $\mathcal{P}(C)$ and the exponential law this would be the same as having a map $C^{op} \times C \to \mathcal{S}$. Since $\mathcal{S} = \mathbf{N}(\mathsf{Kan})$, by the adjunction $\mathfrak{C} \dashv \mathbf{N}$ it is enough to construct a simplicial functor $\mathfrak{C}(C^{op} \times C) \to \mathsf{Kan}$. To get this consider the map $u : \mathfrak{C}(C^{op} \times C) \to \mathfrak{C}(C^{op}) \times \mathfrak{C}(C)$ given by the universal property of the product. This map can be composed with

$$\begin{aligned} \mathfrak{C}(K)^{op} \times \mathfrak{C}(K) &\to \mathsf{Kan} \\ x, y &\mapsto (\hom_{\mathfrak{C}(K)}(x,y))^{\sim} \end{aligned}$$

where the tilde indicates that we are taking a fibrant replacement in the Quillen model structure (for example we can use Ex^{∞} or the singular complex of the geometric realization of $\hom_{\mathfrak{C}(K)}(x,y)$). This gets us a map $Y_C : \mathfrak{C}(C^{op} \times C) \to \mathsf{Kan}$.

Definition 1 (Lurie)**.** By the exponential law and the adjunction $\mathfrak{C} \dashv \mathbf{N}$ the functor Y_C corresponds to a simplicial map $\mathcal{Y}_C : C \to \mathcal{P}(C)$, the *Yoneda embedding* of C.

Note that the above construction would be greatly simplified if $\mathfrak{C}$ preserved products, but it's a simple exercise to show that it doesn't. We have the expected statement.

Proposition 2 ([Lur09, 5.1.3.1])**.** *For any simplicial set K the map $\mathcal{Y}_K : K \to \mathcal{P}(K)$ is fully faithful.*
■

We now give a more conceptual approach to the theory of presheaves in the context of quasi-categories. Consider the functor $Q : \mathsf{sSet}^{op} \to \mathsf{Cat}$ that maps a simplicial set B to the homotopy category of the covariant model structure on sSet/B. Then Q is (almost) representable by $\mathcal{S}$ in the following sense (the only reason why it is not really representable is because $\mathcal{S}$ is not small).

Proposition 3. *The category $Q(B)$ is naturally equivalent to the homotopy category of the quasicategory $\mathcal{S}^B$.* ■

Let us explain the previous proposition in more detail. Recall that the fibrant objects in the covariant model structure on sSet/B are exactly left fibrations. The proposition asserts that there exists a universal left fibration over $\mathcal{S}$. The total space of this fibration is given by *the quasicategory of pointed spaces* defined as the slice quasicategory $\mathcal{S}_\bullet := 1/\mathcal{S}$. The universality in this context means that for any left fibration $E \to B$ there exists a homotopy pullback in sSet_J:

$$\begin{array}{ccc} E & \xrightarrow{f_0} & \mathcal{S}_\bullet \\ {\scriptstyle p}\downarrow & & \downarrow \\ B & \xrightarrow{f_1} & \mathcal{S} \end{array}$$

where (f_0, f_1) are unique, in the sense that the Kan complex of such maps is contractible. The map f_1 is called the *classifying map* of the left fibration p.

We use the proposition to define the Yoneda embedding $C \to \mathcal{P}(C)$: we first construct a left fibration over $C^{op} \times C$ and then consider the classifying map $C^{op} \times C \to \mathcal{S}$.

Definition 4. For any simplicial set define the *twisted diagonal* as the simplicial set C^δ with n-simplices given by:

$$(C^\delta)_n := \hom_{\mathsf{sSet}}((\Delta^n)^{op} * \Delta^n, C).$$

Example 5. Notice that the twisted diagonal of the nerve of a 1-category $\mathcal{C}$ is the nerve of the category of elements of the hom functor $\hom_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$.

Notice that the twisted diagonal comes with a natural projection $(s,t) : C^\delta \to C^{op} \times C$ induced by the inclusions $(\Delta^n)^{op} \hookrightarrow (\Delta^n)^{op} * \Delta^n \hookleftarrow \Delta^n$. By [Joy08, 14.23], the projection (s,t) is a left fibration, so we can consider its classifying map that we denote by $\hom_C : C^{op} \times C \to \mathcal{S}$.

Definition 6 (Joyal)**.** The *Yoneda embedding* is the transpose of the classifying map $\hom_C : C^{op} \times C \to \mathcal{S}$.

The equivalence between the two definitions of the Yoneda embedding is proven in [Lur12, 5.2.1.11].

References

[Joy08] André Joyal. Notes on quasi-categories. *unpublished manuscript*, 2008.

[Joy09] André Joyal. The theory of quasi-categories and its applications. *unpublished manuscript*, 2009.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[Lur12] Jacob Lurie. Higher algebra. avaible from author's website, 2012.