Locally persistent categories and metric properties of interleaving distances

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Outline:

- Motivation: interleavings and homotopy equivalences.
- Locally persistent categories.
- Applications and open question.

Motivation

Fact: $\mathbf{Top}^{\mathbf{R}} \xrightarrow{H_n} \mathbf{Vec}^{\mathbf{R}}$ is stable wrt $d_I^{\mathbf{Top}^{\mathbf{R}}}$ and $d_I^{\mathbf{Vec}^{\mathbf{R}}}$.
(Bubenik, Scott): follows from functoriality.

Problem: $\mathbf{Met} \xrightarrow{\mathrm{VR}} \mathbf{Top}^{\mathbf{R}}$ is not stable wrt d_{GH} and $d_I^{\mathbf{Top}^{\mathbf{R}}}$.

But the composite $\mathbf{Met} \xrightarrow{\mathrm{VR}} \mathbf{Top}^{\mathbf{R}} \xrightarrow{H_n} \mathbf{Vec}^{\mathbf{R}}$ is stable wrt d_{GH} and $d_I^{\mathbf{Vec}^{\mathbf{R}}}$.
(Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot)

Same for the composite $\mathbf{Met} \xrightarrow{\mathrm{VR}} \mathbf{Top}^{\mathbf{R}} \xrightarrow{\pi_0} \mathbf{Set}^{\mathbf{R}}$.

What is special about H_n and π_0? They are homotopy invariant.

Solution: Force $d_I^{\mathbf{Top}^{\mathbf{R}}}$ to be homotopy invariant ($x \simeq y \Rightarrow d(x, y) = 0$).

Many ways to make an interleaving distance homotopy invariant...
(Blumberg, Lesnick), (Frosini, Landi, Mémoli), (Kashiwara, Schapira)
...and many contexts!

Step 1: A general notion of interleaving distance

Idea: If an interleaving is like an *approximate isomorphism*, then it should consist of two “inverse” *approximate morphisms*.

For each δ, ought to have a set of δ-approximate morphisms $\mathrm{Hom}(x, y)_\delta$.

So $\mathrm{Hom}(x, y) : \mathbf{R}_+ \to \mathbf{Set}$ should be a persistent set!

Definition: A **locally persistent category** is a category enriched in persistent sets.

Example: $\mathbf{Top}^\mathbf{R}, \mathbf{Vec}^\mathbf{R}, \mathbf{Set}^\mathbf{R}$ are locally persistent categories with

$$\mathrm{Hom}(x, y)_\delta = \{ \text{ natural transformations that shift degree up by } \delta \ \}.$$

Locally persistent categories

Definition: A **locally persistent category** is a category enriched in persistent sets. Concretely, $\mathcal{C} \in$ **lpCat** consists of:

- a collection of objects $\mathrm{obj}(\mathcal{C})$;
- for $x, y \in \mathcal{C}$, $r \in \mathbf{R}_+$, a set of ***r*-approximate morphisms**

$$\mathrm{Hom}_{\mathcal{C}}(x, y)_r;$$

- for $x \in \mathcal{C}$, an **identity morphism** $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)_0$;
- for $x, y, z \in \mathcal{C}$, $r, s \in \mathbf{R}_+$, a **composition operation**

$$\mathrm{Hom}_{\mathcal{C}}(x, y)_r \times \mathrm{Hom}_{\mathcal{C}}(y, z)_s \to \mathrm{Hom}_{\mathcal{C}}(x, z)_{r+s};$$

- for $x, y \in \mathcal{C}$, $r \leq s \in \mathbf{R}_+$, a **shift operation**

$$\mathbf{S}_{r,s} : \mathrm{Hom}_{\mathcal{C}}(x, y)_r \to \mathrm{Hom}_{\mathcal{C}}(x, y)_s;$$

S.t. unitality, associativity, $\mathbf{S}$ is functorial, $\circ$ and $\mathbf{S}$ are compatible.

Interleaving distance

Let $\mathcal{C} \in \mathbf{IpCat}$, $x, y \in \mathcal{C}$, $\delta \in \mathbf{R}_+$.

Defn: A **δ-interleaving** between x and y consists of $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)_\delta$, $g \in \mathrm{Hom}_{\mathcal{C}}(y,x)_\delta$ such that $g \circ f = \mathbf{S}_{0,2\delta}(\mathrm{id}_x)$ and $f \circ g = \mathbf{S}_{0,2\delta}(\mathrm{id}_y)$.

Defn: The **interleaving distance** of $\mathcal{C}$ is

$$d_I^{\mathcal{C}}(x,y) := \inf\{\delta \,:\, x \text{ and } y \text{ are } \delta\text{-interleaved}\}$$

Recovers the usual interleaving distance of $\mathbf{Top}^{\mathbf{R}}, \mathbf{Vec}^{\mathbf{R}}, \mathbf{Set}^{\mathbf{R}}$.

Question: If $\mathcal{C} \in \mathbf{IpCat}$ has notion of homotopy equivalence, how to make $d_I^{\mathcal{C}}$ homotopy invariant?

Step 2: interleaving distance up to homotopy

Suppose have $\mathcal{C} \in \mathbf{lpCat}$ and $\mathcal{W} \subseteq \mathcal{C}_0$ class of morphisms of $\mathcal{C}$.
(Think of $\mathcal{W}$ as "homotopy equivalences")
Write $x \simeq y$ if x and y are connected by morphisms in $\mathcal{W}$.

Defn: A distance d on obj($\mathcal{C}$) is **$\boldsymbol{\mathcal{W}}$-invariant** if $x \simeq y \Rightarrow d(x, y) = 0$.

Question: How to make $d_I^{\mathcal{C}}$ $\mathcal{W}$-invariant?

Two options:

1. **Quotient interleaving distance** $d_{QI}^{\mathcal{C}} := \left(d_I^{\mathcal{C}}\right)_{/\mathcal{W}}$.

   Largest distance $d \leq d_I^{\mathcal{C}}$ that is $\mathcal{W}$-invariant.

2. **Interleaving distance in the quotient category** $d_{IQ}^{\mathcal{C}} := d_I^{\mathcal{C}[\mathcal{W}^{-1}]}$.

   Localize $\mathcal{C}$ at $\mathcal{W}$, get $\mathcal{C}[\mathcal{W}^{-1}] \in \mathbf{lpCat}$, and use the interleaving distance of $\mathcal{C}[\mathcal{W}^{-1}]$.

Some applications

- Lots of examples of **IpCats**
  - Categories with a flow (Munch, de Silva, Stefanou).
  - Generalized Persistence Modules (Bubenik, de Silva, Scott).
  - Homotopy Interleaving distance [BL] is d_{QI}.
  - Gromov–Hausdorff distance is d_{QI}.
- General stability theorems: Any functor $\mathcal{C} \to \mathcal{D} \in$ **IpCat** is stable.
  - Stability of VR.
  - Stability of density-based filtrations (degree-Rips, kernel density filtration).
- General theorems for metric properties, such as:
  "If $\mathcal{C} \in$ **IpCat** is complete, $\mathcal{W}$ is stable under pullback and powers and closed under sequential limits, then $d_{QI}^{\mathcal{C}}$ is intrinsic and complete."
  - Interleaving distance on tame persistent sets is geodesic and complete.

Open question

Recall: If $\mathcal{C} \in \mathbf{lpCat}$ with "homotopy equivalences" $\mathcal{W} \subseteq \mathcal{C}_0$. Then

$$d_{QI}^{\mathcal{C}} := \left(d_I^{\mathcal{C}}\right)_{/\mathcal{W}} \quad \text{and} \quad d_{IQ}^{\mathcal{C}} := d_I^{\mathcal{C}[\mathcal{W}^{-1}]}$$

(Blumberg, Lesnick) use a quotient interleaving distance (d_{QI}).

(Frosini, Landi, Mémoli), (Kashiwara, Schapira) use interleaving distances in a quotient category (d_{IQ}).

Question: Besides $d_{QI}^{\mathcal{C}} \geq d_{IQ}^{\mathcal{C}}$, what else can be said?

Thm (Lanari, S.): If $\mathcal{C} = \mathcal{M}^{\mathbf{R}}$, with $\mathcal{M}$ cofibrantly generated model category, then $5d_{IQ}^{\mathcal{C}} \geq d_{QI}^{\mathcal{C}}$.

Use this to relate interleavings in homotopy groups to d_{IQ}, which implies a persistent Whitehead theorem (conjectured by [BL]).

- A. J. Blumberg, M. Lesnick
  Universality of the Homotopy Interleaving distance
- P. Frosini, C. Landi, F. Mémoli
  The Persistent Homotopy Type distance
- M. Kashiwara, P. Schapira
  Persistent homology and microlocal sheaf theory
- E. Lanari, L. S.
  Rectification of interleavings and a persistent Whitehead theorem
- L. S.
  Locally persistent categories and metric properties of interleaving distances

Thank you for your attention!