Localization in Homotopy Type Theory

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Background

Definition

Given $k : \mathbb{Z}$ and an abelian group G, the **localization away from k of G** *is a map $G \to G'$ such that*

> *G' is uniquely k-divisible*
>
> *and the map is initial among maps to uniquely k-divisible groups.*

Background

We build on the general framework of:
Modalities in Homotopy Type Theory – Rijke, Shulman, Spitters.

Definition

A **reflective subuniverse** *L is*

a family $\mathsf{isLocal_L} : \mathcal{U} \to \mathsf{Prop}$

together with a reflector $\mathsf{L} : \mathcal{U} \to \mathcal{U}$

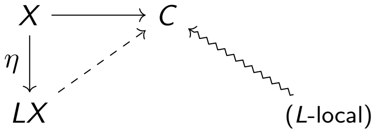
such that $\mathsf{isLocal_L}(LX)$

and for every type X*, a unit* $\eta : X \to LX$

that is initial among maps to L-local types.

Background

The universal property of reflective subuniverses can be interpreted as nondependent elimination:



Example

n-truncation is a reflective subuniverse.

Background

Definition

*Given $f : A \to B$, a type X is f-**local** if $(B \to X) \xrightarrow{f^*} (A \to X)$ is an equivalence.*

Theorem (RSS)

For any map f, the f-local types form a reflective subuniverse.

Example

Localization at $\deg(k) : \mathbb{S}^1 \to \mathbb{S}^1$ is a reflective subuniverse.

$$b \mapsto b$$

$$l \mapsto l^k$$

Goal

Theorem

deg(k)-*localization of a pointed simply connected type localizes all of the homotopy groups.*

Proof.

By localizing the fiber sequences

$$K(\pi_{n+1}(X), n+1) \hookrightarrow \left\|X\right\|_{n+1} \to \left\|X\right\|_{n}$$

to obtain

$$K(L_k\pi_{n+1}(X), n+1) \hookrightarrow \left\|L_{\deg(k)}X\right\|_{n+1} \to \left\|L_{\deg(k)}X\right\|_{n}$$

□

We need...

For simply connected types:

- ☐ deg(k)-localization preserves fiber sequences.
- ☐ A deg(k)-localization of an Eilenberg-Mac Lane space is an Eilenberg-Mac Lane space of the group localized away from k.
- ☐ deg(k)-localization commutes with truncations.

Separated types

Definition

*Given a reflective subuniverse L, a type is L-**separated** if all its identity types are L-local.*

Example

If L is n-truncation, then the L-separated types are the $(n+1)$-truncated types.

Example

For a map $f : A \to B$, a type is f-separated precisely if it is Σf-local (where $\Sigma f : \Sigma A \to \Sigma B$).

Theorem

L-separated types form a reflective subuniverse $\mathbf{L}'$.

Localization and fiber sequences

Theorem (dependent elimination for L')

Let $P : L'X \to \mathcal{U}$ be a type family **with L-local fibers**. *To prove* $\prod_{x:L'X} P(x)$ *it is enough to prove* $\prod_{x:X} P(\eta x)$.

Theorem

Given a pointed type X we have $\Omega L'X \simeq L\Omega X$.

In particular, for any $f : A \to B$ we have $\Omega L_{\Sigma f} X \simeq L_f \Omega X$.

When $L = L_{\deg(k)}$ and X is simply connected, $LX = L'X$.

We need. . .

For simply connected types:

- ☑ deg(k)-localization preserves fiber sequences.
- ☐ A deg(k)-localization of an Eilenberg-Mac Lane space is an Eilenberg-Mac Lane space of the group localized away from k.
- ☐ deg(k)-localization commutes with truncations.

Localization of loop spaces

Theorem

Let X be pointed and simply connected. Then the $\deg(k)$*-localization of ΩX is equivalent to the colimit of the sequence*

$$\Omega X \xrightarrow{k} \Omega X \xrightarrow{k} \cdots$$

Example

When ΩX is an Eilenberg-Mac Lane space of an abelian group.

We need. . .

For simply connected types:

- ☑ $\deg(k)$-localization preserves fiber sequences.
- ☑ A $\deg(k)$-localization of an Eilenberg-Mac Lane space is an Eilenberg-Mac Lane space of the group localized away from k.
- ☐ $\deg(k)$-localization commutes with truncations.

Localization and truncation

Since $\deg(k)$ preserves Eilenberg-Mac Lane spaces, it follows by induction that it preserves truncatedness.

Theorem

Truncation preserves $\deg(k)$*-localness.*

Proof.

From the formula $\Omega\|X\|_{n+1} \simeq \|\Omega X\|_n$ and naturality. □

Corollary

$\deg(k)$*-localization commutes with truncations.*

We need...

For simply connected types:

- ☑ deg(k)-localization preserves fiber sequences.
- ☑ A deg(k)-localization of an Eilenberg-Mac Lane space is an Eilenberg-Mac Lane space of the group localized away from k.
- ☑ deg(k)-localization commutes with truncations.

We have proved that $\deg(k)$-localization localizes the homotopy groups of a simply connected type.

We also consider localizations away from *sets* of numbers.

And we have results about the interaction between localizations at sets of numbers and connectedness, truncations, and other localizations.

Thanks for listening!