Stable quasicategories

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We will start by introducing the notion of stable quasicategory and related concepts. Then we will show (without too many details) that the homotopy category of a stable quasicategory is triangulated. We will finish by proving some basic closure properties of stable quasicategories.

The references are:

- [Lur12] for the main content;
- [Har13] for alternative proofs;
- [Lur09] for background on quasicategories.

Let us start by introducing the loop functor, the suspension functor, fibers and cofibers. Throughout the talk $\mathcal{C}$ will be a quasicategory.

Definition 1. A *zero object* in $\mathcal{C}$ is an initial and terminal object. Zero objects are denoted by $0 \in \mathcal{C}$. A quasicategory that admits a zero object is called *pointed*. A functor that preserves zero objects is called *reduced*.

Notice that if $\mathcal{C}$ is pointed then it has a unique zero object up to contractible choice. If $\mathcal{C}$ is pointed, given $X, Y \in \mathcal{C}$ consider the composition:

$$\mathrm{Map}_{\mathcal{C}}(X, 0) \times \mathrm{Map}_{\mathcal{C}}(0, Y) \xrightarrow{\circ} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

The domain of this map is contractible so this determines a map $0 : X \to Y$ up to contractible choice, the *zero map*.

Definition 2. Let $\mathcal{C}$ be pointed.

- A *triangle* in $\mathcal{C}$ is a diagram of the form:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

- A *fiber sequence* is a triangle that is a pullback.

- A *cofiber sequence* is triangle that is a pushout.

Definition 3. Let $\mathcal{C}$ be pointed. Given $f : y \to z$ in $\mathcal{C}$, a *fiber* for f is a fiber sequence:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Z \end{array}$$

Dually one defines the notion of *cofiber*.

If a pointed quasicategory admits fibers we clearly can construct a fiber for each map, but the question is whether we can do this functorially.

Lemma 4 (cf. [Lur09, Proposition 4.3.2.15]). *Let K be a simplicial set and $\mathcal{D}$ quasicategory that admits K-shaped limits. Call $i : K \hookrightarrow K^{\triangleleft}$ to the natural inclusion and let $\mathrm{Fun}^{univ}(K^{\triangleleft}, \mathcal{C}) \subseteq \mathrm{Fun}(K^{\triangleleft}, \mathcal{C})$ be the full subquasicategory spanned by universal cones. Then the restriction $\mathrm{Fun}^{univ}(K^{\triangleleft}, \mathcal{C}) \xrightarrow{i^*} \mathrm{Fun}(K, \mathcal{C})$ is a trivial fibration.* ■

This means that in a quasicategory that admits limits, limits can be chosen functorially, and there is a unique way of doing this, up to contractible choice.

We are ready to introduce the loop functor. Let $\mathcal{C}$ be pointed. Let $\mathcal{M}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ denote the full subquasicategory spanned by pullbacks of the form:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

There are projections:

$$\mathcal{C} \xleftarrow{d_{0,0}} \mathcal{M}^{\Omega} \xrightarrow{d_{1,1}} \mathcal{C}$$

given by projecting the upper left vertex and the lower right vertex respectivelly.

If $\mathcal{C}$ admits pullbacks $d_{1,1}$ is a trivial fibration since $\Delta^1 \times \Delta^1 = (\Lambda^2_2)^{\triangleleft}$ and there is a pullback:

$$\begin{array}{ccc} \mathcal{M}^{\Omega} & \longrightarrow & \mathrm{Fun}^{univ}(\Delta^1 \times \Delta^1, \mathcal{C}) \\ d_{1,1} \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathrm{Fun}(\Lambda^2_2, \mathcal{C}) \end{array}$$

$$X \longmapsto (0 \to X \leftarrow 0)$$

where the right vertical map is a trivial fibration by Lemma 4.

Definition 5. Let $\mathcal{C}$ be pointed and admit pullbacks. The *loop functor* $\Omega : \mathcal{C} \to \mathcal{C}$ is defined as the composition $d_{0,0}$ with a section of $d_{1,1}$. Dually, if $\mathcal{C}$ is pointed and admits pushouts we define the *suspension functor* $\Sigma : \mathcal{C} \to \mathcal{C}$.

The two constructions are related.

Proposition 6. *If $\mathcal{C}$ is pointed and it admits pullbacks and pushouts we have $\Sigma \vdash \Omega$.*

Proof. By the universal property of suspensions, maps $\Sigma X \to Y$ correspond to squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

which in turn correspond to maps $X \to \Omega Y$. ■

We can carry out a similar construction that lets us construct fibers and cofibers functorially. Let $\mathcal{C}$ be pointed and let $\mathcal{E} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the full subquasicategory spanned by fiber sequences. Consider the projection $\mathcal{E} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ defined by sending a fiber sequence

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Z \end{array}$$

to the arrow $Y \xrightarrow{f} Z$. If $\mathcal{C}$ admits pullbacks then this projection is a trivial fibration since we can form the pullback:

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathrm{Fun}^{univ}(\Delta^1 \times \Delta^1, \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^1, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\Lambda^2_2, \mathcal{C}) \end{array}$$

$$(Y \xrightarrow{f} Z) \longmapsto (0 \to Z \xleftarrow{f} Y)$$

Definition 7. Let $\mathcal{C}$ be pointed and admit pullbacks. The functor $\mathsf{fib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ is defined as a section of the projection above. Dually, if $\mathcal{C}$ is pointed and admits cofibers we define the functor $\mathsf{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$.

The composition $\mathrm{Fun}(\Delta^1, \mathcal{C}) \xrightarrow{\mathsf{fib}} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \xrightarrow{d_{0,0}} \mathcal{C}$. will also be called fib. Dually we have $\mathsf{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$.

Notice that we have an adjunction:

$$X \longmapsto (0 \to X)$$

$$\mathcal{C} \rightleftarrows \operatorname{Fun}(\Delta^1, \mathcal{C})$$

$$\text{fib}$$

This implies:

Corollary 8. *The functor fib* : $\operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ *preserves limits. Dually, functor cofib* : $\operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ *preserves colimits.*

Definition 9. A quasicategory $\mathcal{C}$ is *stable* if:

- it is pointed;
- it admits fibers and cofibers;
- a triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

Notice that being stable is a property and not extra structure. Also being stable is a selfdual notion.

Definition 10. A functor between stable quasicategories is *exact* if:

- it is reduced;
- it preserves fiber sequences or cofiber sequences (and hence both).

The fact that in a stable quasicategory a triangle is a fiber sequence if and only if it is a cofiber sequence directly implies:

Proposition 11. *In a stable quasicategory the adjunction* $\Sigma \dashv \Omega$ *forms an adjoint equivalence.* ■

Example 12. The various categories of spectra with their respective model structures introduced last talk present a stable quasicategory that will be presented in a future talk.

Example 13. Given a nice enough abelian category, its derived category is the homotopy category of a stable quasicategory that will also be presented in the future.

We know that the homotopy category of spectra is triangulated, and this also holds for the derived category of an abelian category. This is no coincidence: the homotopy category of a stable quasicategory is cannonically triangulated. Our next goal is to prove this claim.

Recall that an *additive category* is a category enriched over Ab that admits finite products or coproducts (and a fortiori both). Recall also that in an additive category the natural map $X \amalg Y \to X \times Y$ is an isomorphism, and thus one usually writes $X \oplus Y := X \times Y \simeq X \amalg Y$.

Definition 14. A *triangulated category* is given by:

1. an additive category $\mathcal{D}$;

2. a translation functor

$$\begin{aligned} \mathcal{D} &\to \mathcal{D} \\ X &\mapsto X[1] \end{aligned}$$

which an equivalence;

3. a collection of sequences $X \to Y \to Z \to X[1]$ called *distinguished triangles*.

Such that a list of axioms is verified.

In our context the axioms are straightforward to verify, so for the sake of brevity we just mention a few:

(TR1)(a) Every arrow $X \xrightarrow{f} Y$ can be extended to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$.

(TR3) Given a morphism between arrows (i.e. a commutative square) and extensions of both arrows to distinguished triangles, the commutative square can be extended to a morphism of distringuished triangles.

We will need the quasicategorical analoge of the the two pullback lemma.

Lemma 15 ([Lur09, Lemma 4.4.2.1]). *Given a (homotopy commutative) diagram of the form:*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

Suppose that the right square is a pullback. Then the left square is a pullback if and only if the outer square is a pullback. ■

Lemma 16. *A stable quasicategory admits pullbacks and pushouts.*

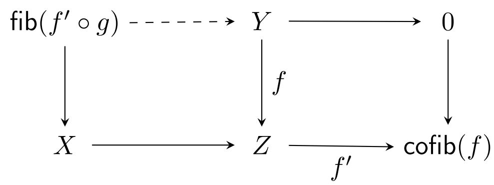
Proof. By symmetry it is enough to show it admits pullbacks. So assume given a diagram:

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ X & \xrightarrow[g]{} & Z \end{array}$$

Construct the cofiber of f to obtain:

$$\begin{array}{ccccc} & & Y & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \\ X & \longrightarrow & Z & \xrightarrow[f']{} & \mathsf{cofib}(f) \end{array}$$

where the right square is both a pushout and a pullback. Now construct the fiber of $f' \circ g$ which gives us:



where the dashed arrow exists by the (limit) universal property of the right square. Since the outer square is a pullback by definition, the two pullback lemma implies that the left square is a pullback as needed. ■

Corollary 17. *If $\mathcal{C}$ is stable then* $\mathsf{Ho}(\mathcal{C})$ *admits finite products and coproducts.*

Proof. Start by noticing that $\mathcal{C}$ has finite products (coproducts) since it has terminal (initial) object and pullbacks (pushouts). A quick check shows that products (coproducts) in $\mathcal{C}$ provide products (coproducts) for $\mathsf{Ho}(\mathcal{C})$. ■

On the other hand, we have:

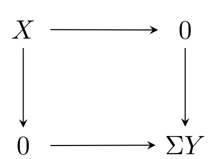
Proposition 18. *If $\mathcal{C}$ is stable, then* $\mathsf{Ho}(\mathcal{C})$ *is cannonically enriched over* Ab.

Proof. We start with an abstract proof:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Ho}(\mathcal{C})}(X, Y) &\simeq \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \\ &\simeq \pi_0 \operatorname{Map}_{\mathcal{C}}(X, \Omega^2 \Sigma^2 Y) \\ &\simeq \pi_0 \Omega^2 \operatorname{Map}_{\mathcal{C}}(X, \Sigma^2 Y) \end{aligned}$$

Since the last object is an abelian group and the isomorphisms are natural and bilinear with respect to composition, we are done.

Let us also give an explicit construction. We define an "addition" operation in $\operatorname{Map}_{\mathcal{C}}(X, Y)$. First observe that having an arrow $X \to Y$ is equivalent to having an arrow $X \to \Omega\Sigma Y$, which in turn is equivalent to having a (homotopy) commutative diagram



Now suppose $f_1, f_2 : X \to Y$ are classfied by two diagrams as above, and call h_1, h_2 to the homotopies rendering commutative the corresponding diagram. Then let $f_1 + f_2$ be classified by a diagram as above where the homotopy rendering it commutative is given by the composite homotopy $h_1 \bullet h_2$.

The result follows by checking that the construction verifies the following properties:

- It is associative up to higher homotopy;

- it has inverses given by reflecting the squares up to higher cells;
- it has units given by the zero map;
- it is bilinear with respect to composition;
- the dual construction using suspensions instead of loops defines another operation compatible with the above. The Eckmann-Hilton argument implies that the operations are commutative. ■

Using the two last results we deduce the following.

Proposition 19. *If $\mathcal{C}$ is stable then $\mathsf{Ho}(\mathcal{C})$ is additive, and for every $X, Y \in \mathcal{C}$ the natural map $X \amalg Y \to X \times Y$ is an equivalence.*

We are ready for the main result of this section.

Theorem 20. *If $\mathcal{C}$ is stable then $\mathsf{Ho}(\mathcal{C})$ is cannonically triangulated.*

Proof. We give the structure and check the two axioms we mentioned. By Proposition 19 we have (1) $\mathsf{Ho}(\mathcal{C})$ is additive. For (2) take the translation to be Σ, which we know it is an equivalence by Proposition 11. Finally a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ in $\mathsf{Ho}(\mathcal{C})$ is a distinguished triangle if there exists a diagram in $\mathcal{C}$:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow \\ 0 & \longrightarrow & Z & \xrightarrow[\tilde{h}]{} & W \end{array}$$

such that:

- $\tilde{f}$ is a lift of f;
- $\tilde{g}$ is a lift of g;
- $\tilde{h}$ composed with the map $W \to X[1]$ induced by the universal property of $X[1]$ is a lift of h.

To prove that **(TR1)(a)** holds we use the existence of cofibers (twice) to extend an arrow to a diagram as above. To prove that **(TR3)** holds we use the universal property of cofibers to extend a commutative square to a morphism of distinguished triangles. ■

We conclude with some basic closure properties of stable quasicategories.

Proposition 21. *If $\mathcal{C}$ is stable and K is a simplicial set the quasicategory $\mathrm{Fun}(K, \mathcal{C})$ is stable.* ■

Proof. In functor quasicategories, limits and colimits are computed pointwise. ■

Proposition 22. *Let $\mathcal{C}$ be pointed. Then $\mathcal{C}$ is stable if and only if:*

- *it admits finite limits and colimits;*
- *a square is a pushout if and only if it is a pullback.*

Proof. The "if" implication is immediate. For the "only if" we already know that $\mathcal{C}$ admits pullbacks and since we have initial object, it also admits products (as we already argued). The existence of finite limits now follows from [Lur09, Corollary 4.4.2.4], which says that admitting finite limits is equivalent to admitting finite products and pullbacks. By duality $\mathcal{C}$ also admits finite colimits.

To prove the second condition, notice that by duality it is enough to show that a pushout is also a pullback. Suppose given a pushout

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & P \end{array}$$

and construct the cofiber of g:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \xrightarrow[g]{} & P \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathsf{cofib}(g) \end{array}$$

Since both squares are pushouts Lemma 15 implies that the outer square is a pushout. Since both the outer square and the lower square are cofiber sequences they are also fiber sequences, since $\mathcal{C}$ is stable. So using Lemma 15 again (but in the other direction) we deduce that the upper square is a pullback. ∎

A very similar argument proves:

Proposition 23. *A functor between stable quasicategories is exact if and only if it preserves finite limits (or colimits).*

References

[Har13] Yonatan Harpaz. Introduction to stable ∞-categories, 2013.

[Lur09] Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[Lur12] Jacob Lurie. Higher algebra. avaible from author's website, 2012.