Homotopy Interleavings

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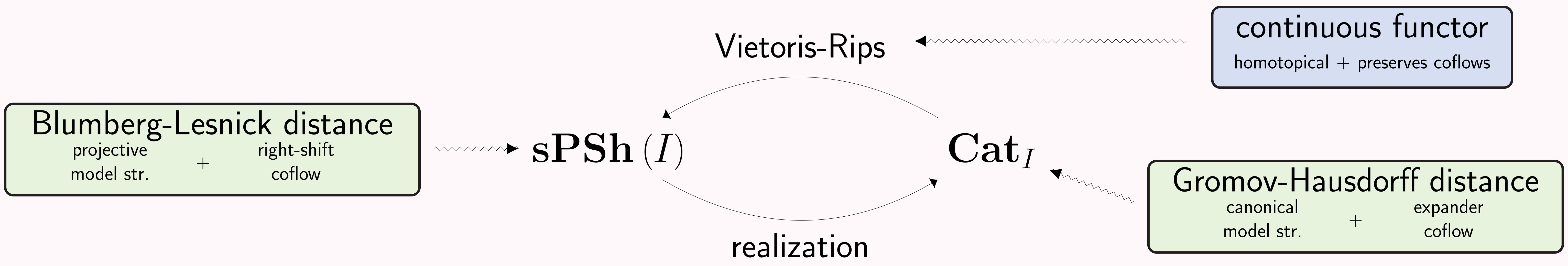
We combine Blumberg-Lesnick's homotopy interleaving distance [BL17] with the theory of interleavings in categories with a flow of de Silva, Munch, Stefanou [DSMS18], and generalize the framework to a theory of homotopy interleavings in $\mathbf{PSh}(I)$-enriched model categories, for I a suitable monoidal poset.
In this framework, the Gromov-Hausdorff distance is a homotopy interleaving distance, and the continuity of the Vietoris-Rips functor follows from a few abstract arguments.

Definition 1. Let I be a suitable monoidal poset, such as $([0,\infty], \geq, +)$. Let
- **metric spaces** be $\mathbf{Cat}_I$ (categories enriched over I);
- **persistent sets** be $\mathbf{PSh}(I)$ (presheaves over I);
- **persistent spaces** be $\mathbf{sPSh}(I)$ (simplicial presheaves over I).

By nerve-realization we get the **Vietoris-Rips adjunction**:



1 Interleavings and Homotopy Interleavings

Day convolution endows the category $\mathbf{PSh}(I)$ with a monoidal structure, with Yoneda $\mathcal{Y} : I \to \mathbf{PSh}(I)$ strong monoidal.

Definition 2. For $\delta \in I$, let I^δ be the $\mathbf{PSh}(I)$-enriched category with objects $\{0,1\}$, and $\mathbf{hom}(0,1) = \mathbf{hom}(1,0) = \mathcal{Y}(\delta)$, $\mathbf{hom}(0,0) = \mathbf{hom}(1,1) = \mathcal{Y}(0)$.

Definition 3. Let $\mathcal{C}$ be $\mathbf{PSh}(I)$-enriched. A ***δ*-interleaving** between $x, y \in \mathcal{C}$ is a $\mathbf{PSh}(I)$-enriched functor $f : I^\delta \to \mathcal{C}$ with $f(0) = x$ and $f(1) = y$.

Theorem 4 (S., c.f. [DSMS18, Thm. 2.3], [BM13, Sec. 3.10])**.** In any $\mathbf{PSh}(I)$-enriched category, interleavings compose.

This means that if x, y are ϵ-interleaved, and y, z are δ-interleaved, then x, z are $(\epsilon + \delta)$-interleaved. So the **strong interleaving distance**,

$$d_{SI}(x,y) = \inf_{\delta \in I^{op}} \{x \text{ and } y \text{ are } \delta\text{-interleaved}\},$$

satisfies the triangle inequality.

Definition 5. Let $\mathcal{M}$ be $\mathbf{PSh}(I)$-enriched with a model structure.
A ***δ*-homotopy interleaving** between $x, y \in \mathcal{M}$ is given by $x \simeq x'$, $y \simeq y'$, and a δ-interleaving between x' and y'.

Theorem 6 (S., c.f. [BL17, Section 4])**.** In a tensored and cotensored $\mathbf{PSh}(I)$-enriched category with a model structure and an enriched fibrant replacement, such that cotensoring by representables preserves trivial fibrations, homotopy interleavings compose.

In the situation above, the **homotopy interleaving distance**,

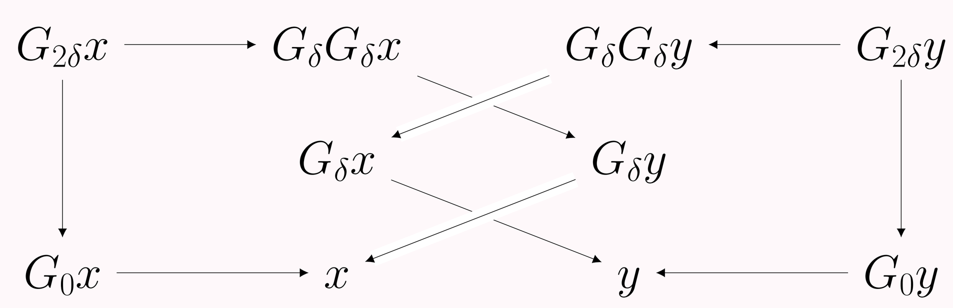
$$d_{HI}(x,y) = \inf_{\delta \in I^{op}} \{x \text{ and } y \text{ are } \delta\text{-homotopy interleaved}\},$$

satisfies the triangle inequality.

2 (Co)flows

Definition 7 ([DSMS18, Def. 2.1])**.** A **coflow** on a category $\mathcal{C}$ is a lax monoidal functor $G : I \to \mathbf{End}(\mathcal{C})$.

Theorem 8. A coflow induces a $\mathbf{PSh}(I)$-enrichment.



A δ-interleaving in a category with a coflow (c.f. [DSMS18]).

Definition 9 (Right-shift coflow)**.** There is a coflow on $\mathbf{sPSh}(I)$ given by mapping $r \in I$ to the functor that "shifts the presheaf to the right".

Definition 10 (Expander coflow)**.** There is a coflow on $\mathbf{Cat}_I$, given by mapping $r \in I$ to the functor that adds r to all the homs between distinct objects.

3 Model structures

Theorem 11 (Canonical model structure)**.** There is a model structure on $\mathbf{Cat}_I$ where weak equivalences are the I-enriched equivalences.

Theorem 12 (Projective model structure)**.** There is a model structure on $\mathbf{sPSh}(I)$ where the weak equivalences and the fibrations are the sectionwise weak equivalence and fibrations.

4 Gromov-Hausdorff is a homotopy interleaving distance

The homotopy interleaving distance of [BL17, Def. 3.7] now generalizes to arbitrary I.

Definition 13 (c.f. [BL17, Def. 3.7])**.** The projective model structure together with the right-shift coflow give the **Blumberg-Lesnick distance** on $\mathbf{sPSh}(I)$.

Theorem 14 (S.)**.** When $I = [0,\infty]$, the Gromov-Hausdorff distance between Lawvere spaces [Law73] is twice the homotopy interleaving distance on $\mathbf{Cat}_I$ given by the canonical model structure together with the expander coflow.

Theorem 15 (S.)**.** Vietoris-Rips is homotopical and laxly respects the coflows.

We recover the continuity of Vietoris-Rips [BL17, Prop. 1.5]. For $P, Q \in \mathbf{Cat}_I$,

$$d_{BL}(\mathbf{vr}(P), \mathbf{vr}(Q)) \leq 2 d_{GH}(P,Q).$$

5 Universality

Definition 16. Let $I = [0,\infty]$. Given $X, Y \in \mathbf{sPSh}(I)$ with all transition maps monomorphisms, and such that $X(\infty) = Y(\infty)$, we let

$$d_\infty(X,Y) = \sup_{\sigma \in X(\infty)} \{\text{difference between the times } \sigma \text{ appears in } X \text{ and } Y\}.$$

A distance bounded by d_∞ is **stable**.

Theorem 17 ([BL17, Thm. 1.7])**.** The Blumberg-Lesnick distance is the largest distance that is homotopy invariant and stable.

This is a consequence of the following theorem, together with a characterization of the cofibrant objects of $\mathbf{sPSh}(I)$ (ask me about this!).

Theorem 18 (S.)**.** If $\mathcal{M}$ has an enriched cofibrant replacement, the homotopy interleaving distance is the largest distance that is homotopy invariant and bounded above by d_{SI} on cofibrant objects (stable).

Acknowledgements

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References

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