Homotopy Interleavings: a framework for stability theorems

Luis Scoccola
lscoccol@uwo.ca

University of Western Ontario

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jww Alex Rolle

Outline:

- Give a framework for proving stability results in TDA.
- Apply it to Vietoris-Rips and Degree-Rips.

Idea

Blumberg and Lesnick 2017 introduce the Homotopy Interleaving distance between persistent spaces.

Our starting point is that many other distances in TDA can be seen as homotopy interleaving distances:

- The Gromov-Hausdorff distance between metric spaces
- The Gromov-Hausdorff-Prokhorov distance between metric measure spaces

Main Lemma

A functor between categories with such distances is continuous if it preserves weak equivalences and strong interleavings.

Note that a version of this lemma for strong interleavings has already been considered in, e.g., de Silva et al. 2018.

Strong interleavings

$I = [0, \infty], \quad \delta \in I$

$\mathcal{S} =$ simplicial complexes

Met = extended pseudo metric spaces

Definition: A **$\boldsymbol{\delta}$-strong interleaving** between $P, Q \in$ **Met** is a bijection of sets $f : P \to Q$, such that

$$|d(p,p') - d(f(p), f(p'))| \leq \delta, \ \forall p, p' \in P$$

Definition: A **$\boldsymbol{\delta}$-strong interleaving** between $X, Y \in \mathcal{S}^I$ is given by maps $f : X \to Y[\delta]$, $g : Y \to X[\delta]$, such that

$$\begin{array}{ccccc} X & & & & Y \\ \downarrow & \searrow & & \swarrow & \downarrow \\ X[\delta] & & & & Y[\delta] \\ \downarrow & \searrow & & \swarrow & \downarrow \\ X[2\delta] & & & & Y[2\delta] \end{array}$$

Proposition: The Vietoris-Rips filtration, $VR : \mathbf{Met} \to \mathcal{S}^I$, preserves δ-strong interleavings.

A set-up for strong interleavings

Remark

Given a $\mathbf{PSh}(I)$-enriched category, we can define a notion of strong interleaving.

Proposition

The triangle inequality for strong interleavings holds in any $\mathbf{PSh}(I)$-enriched category.

Notation

If $A, B \in \mathcal{C}$ are δ-strongly interleaved we write $A \underset{\delta}{\rightleftarrows} B$

Weak equivalences

Definition: A **category with weak equivalences** is a category $\mathcal{C}$ together with a class of morphisms W that contains all identities.

Definition: A **weak equivalence** between $P, Q \in \mathbf{Met}$ is
a distance preserving map $f : P \to Q$, such that
for all $q \in Q$, there exists $p \in P$ with $d(f(p), q) = 0$.

Definition: A **weak equivalence** between $X, Y \in \mathcal{S}^I$ is
a natural transformation $\alpha : X \to Y$, such that
$\alpha(r) : X(r) \to Y(r)$ is a homotopy equivalence for every $r \in I$.

Definition: Objects $A, B \in \mathcal{C}$ are **weakly equivalent** if they are connected by a zig-zag of weak equivalences. We write $A \simeq B$.

Proposition: Vietoris-Rips preserves weak equivalences.

Homotopy interleavings

Let $\mathcal{C}$ be a category with weak equivalences and a notion of strong interleaving (such as **Met** or $\mathcal{S}^I$).

Definition

A **δ-homotopy interleaving** *between $A, B \in \mathcal{C}$ is*

$$\begin{array}{ccc} A & & B \\ \sim \big| & & \big| \sim \\ A' & \overset{\delta}{\rightleftarrows} & B' \end{array}$$

Definition

The **homotopy interleaving distance** *is the biggest distance*

$$d_{HI} : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to I$$

such that $d_{HI}(A, B) \leq \delta$ whenever there is a δ-homotopy interleaving between A and B.

Continuity of Vietoris-Rips

Remark

In $\mathcal{S}^I$, the homotopy interleaving distance is the homotopy interleaving distance of Blumberg and Lesnick.

Theorem (S.)

In **Met***, the homotopy interleaving distance coincides with the Gromov-Hausdorff distance (times* $1/2$*).*

So Vietoris-Rips is Lipschitz wrt the Gromov-Hausdorff distance and the homotopy interleaving distance (recovering a result of Blumberg and Lesnick).

This implies, e.g., stability of barcodes.

Degree-Rips

Definition

A **metric measure space** (X, d, μ) *is a metric space* (X, d) *together with a Borel measure* μ*. Denote the category by* **MMet***.*

Definition (c.f. Lesnick, Wright 2015)

The **Degree-Rips filtration**, $DR : \mathbf{MMet} \to \mathcal{S}^{I \times I}$*, of* $X \in \mathbf{MMet}$*:*

$$DR_{e,k}(X) = VR_e(X_{e,k}),$$

where $e, k \in I$ *and*

$$X_{e,k} = \{x \in X \mid \mu(B_e(x)) \geq k\}.$$

GP and GHP distances

Definition

Gromov-Prokhorov distance *between* $X, Y \in \textbf{MMet}$*:*

$$d_{GP}(X, Y) = \inf\{d_P(i_*(\mu_X), j_*(\mu_Y))\},$$

where the infimum is taken over all isometric embeddings $i : X \to Z$, $j : Y \to Z$ *into a common metric space.*

Definition

Gromov-Hausdorff-Prokhorov distance *between* $X, Y \in \textbf{MMet}$*:*

$$d_{GHP}(X, Y) = \inf\{d_H(i(X), j(Y)) \vee d_P(i_*(\mu_X), j_*(\mu_Y))\},$$

where the infimum is taken over all isometric embeddings $i : X \to Z$, $j : Y \to Z$ *into a common metric space.*

Continuity of Degree-Rips

The GHP distance is again a homotopy interleaving distance, and using our framework:

Theorem (Rolle, S.)

The Degree-Rips filtration is Lipschitz with respect to GHP and the homotopy interleaving distance on bi-persistent spaces.

Continuity wrt GP?

Continuity of Degree-Rips

Definition

The **local density bound** $\psi_X : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ *of* $X \in$ **MMet** *is*

$$\psi_X(\epsilon) = \inf_{x \in X} \{\mu_X(B_\epsilon(x))\}.$$

Definition

Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ *and let* $w > 0$*. Let* $\mathbf{MMet}_{f,w}$ *be the class of* $X \in$ **MMet** *such that* $\psi_X \geq f$ *and* $\mu(X) \geq w$*.*

Example

Let $X \in$ **MMet** with bounded measure and metric. Then it is **doubling** if and only if it is in $\mathbf{MMet}_{f,w}$, for f a line through the origin.

Proposition (Rolle, S.)

A sequence in $\mathbf{MMet}_{f,w}$ *converges in GHP if and only if it converges in GP.*

Continuity of Degree-Rips

Theorem (Rolle, S.)
The Degree-Rips filtration is continuous with respect to GP and the homotopy interleaving distance on bi-persistent spaces, when restricted to $\mathbf{MMet}_{f,w}$.

Other possible approaches include work of Blumberg and Lesnick.

Under the same hypothesis of the theorem one can prove:

- Continuity of DR wrt Gromov-Wasserstein distance, $p \geq 1$.
- DR of a finite sample converges to DR of the sampled space.

Future: extend to other filtrations.

Thank you for your attention!