Notes on Gabiel-Ulmer duality

Luis Scoccola

Notation 1. Denote by LEX the 2-category of categories with finite limits and finite limit preserving funtors. We denote the 2-full inclusion of this two category into the 2-category of categories by $i : \mathsf{LEX} \hookrightarrow \mathsf{CAT}$.

Denote by LFC the 2-category of categories with small limits and filtered colimits, and small limits and filtered colimits preserving functors. Similarly we define the 2-full inclusion $j : \mathsf{LFC} \hookrightarrow \mathsf{CAT}$.

Notice that Set lives in both 2-categories, and moreover finite limits, small limits and filtered colimits all commute in Set.

Consider the adjunction:

$$\mathcal{F}_0 : \mathsf{CAT} \rightleftarrows \mathsf{CAT}^{op} : \mathcal{G}_0$$

where both F and G are $\mathsf{CAT}(-, \mathsf{Set})$ and the unit and counit η_0, ε_0 are the evaluation functors:

$$\begin{aligned} \mathcal{C} &\to \mathsf{CAT}(\mathsf{CAT}(\mathcal{C}, \mathsf{Set}), \mathsf{Set}) \\ c &\mapsto (F \mapsto F(c)) \end{aligned}$$

The fact that both small limits and filtered colimits commute with finite limits in Set implies the following facts.

Lemma 2. *Restricting the functors that we consider we have:*

- *If $\mathcal{C}$ has finite limits then $\mathsf{LEX}(\mathcal{C}, \mathsf{Set})$ has limits and filtered colimits, and these are computed pointwise.*
- *If $\mathcal{D}$ has limits and filtered colimits then $\mathsf{LFC}(\mathcal{D}, \mathsf{Set})$ has finite limits, and these are computed pointwise.*

Here "computed pointwise" means that they are preserved by evaluation. Moreover these limits and colimits are preserved by the inclusions $\mathsf{LEX}(\mathcal{C}, \mathsf{Set}) \hookrightarrow \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ and $\mathsf{LFC}(\mathcal{D}, \mathsf{Set}) \hookrightarrow \mathsf{CAT}(\mathcal{D}, \mathsf{Set})$.

Proof. The proofs of the two facts are very similar, so let us prove only the first one. Consider a filtered diagram $F_i \in \mathsf{LEX}(\mathcal{C}, \mathsf{Set})$. This diagram has a colimit in $\mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ that we call F. Since $i : \mathsf{LEX}(\mathcal{C}, \mathsf{Set}) \hookrightarrow \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ is full we only need to prove that F preserves finite limits.

For this we notice that:

$$
\begin{aligned}
F(\lim_j c_j) &= \operatorname{colim}_i F_i(\lim_j c_j) \\
&= \operatorname{colim}_i \lim_j F_i(c_j) \\
&= \lim_j \operatorname{colim}_i F_i(c_j) \\
&= \lim_j F(c_j).
\end{aligned}
$$

The fact that this limit is preserved by the inclusion i is clear by construction. ■

Moreover we have:

- If $\varphi : \mathcal{C} \to \mathcal{C}'$ preserves finite limits then $\varphi^* : \mathsf{LEX}(\mathcal{C}', \mathsf{Set}) \to \mathsf{LEX}(\mathcal{C}, \mathsf{Set})$ preserves limits and filtered colimits.
- If $\psi : \mathcal{D} \to \mathcal{D}'$ preserves limits and filtered colimits then $\psi^* : \mathsf{LFC}(\mathcal{D}', \mathsf{Set}) \to \mathsf{LFC}(\mathcal{D}, \mathsf{Set})$ preserves finite limits.

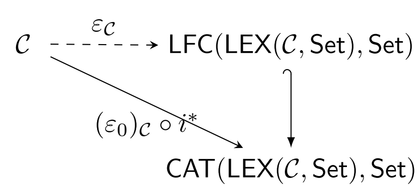
This follows from the fact that limits and colimits in the functor categories are computed pointwise. The above implies that we can refine the previous adjunction to an adjunction:

$$F : \mathsf{LFC} \rightleftarrows \mathsf{LEX}^{op} : G$$

where $F = \mathsf{LFC}(-, \mathsf{Set})$ and $G = \mathsf{LEX}(-, \mathsf{Set})$. To see that this is indeed an adjunction we must define the unit and the counit η and ε. Consider $i^* : \mathsf{CAT}(\mathsf{CAT}(\mathcal{C}, \mathsf{Set}), \mathsf{Set}) \to \mathsf{CAT}(\mathsf{LEX}(\mathcal{C}, \mathsf{Set}), \mathsf{Set})$ and precompose with $(\varepsilon_0)_{\mathcal{C}} : \mathcal{C} \to \mathsf{CAT}(\mathsf{CAT}(\mathcal{C}, \mathsf{Set}), \mathsf{Set})$, which results in a functor:

$$(\varepsilon_0)_{\mathcal{C}} \circ i^* : \mathcal{C} \to \mathsf{CAT}(\mathsf{LEX}(\mathcal{C}, \mathsf{Set}), \mathsf{Set})$$

which by the above remarks factors:



giving us the counit $\varepsilon_{\mathcal{C}}$. Dually we construct η.

The fact that these natural transformations are natural and are indeed a unit and a counit follows from the functoriality of the construction and the fact that η_0 and ε_0 already were the unit and counit of an adjunction.

Definition 3. Let k be a regular cardinal then:

- A locally small category is *k-accessible* if it has k-directed colimits and there is a (small) set of k-compact objects that generates the category under k-directed colimits.
- A category is *locally k-presentable* if it is k-accessible and has all small colimits.

Notice that since k is regular, a k-accessible category is also k'-accessible for any $k' \geq k$.

Lemma 4. *A category is k-accessible if and only if it is equivalent to* $\mathsf{Ind}_k(S)$ *for some small category S.* ■

In particular a locally small category is *locally finitely presentable* if it has all colimits and has a small set of generators that consists on finitely presentable objects.

Definition 5. The 2-full subcategory $\mathsf{LFP} \hookrightarrow \mathsf{LFC}$ is the 2-category spanned by categories that have a small set of objects that generate the category under filtered colimits.

Concretely the objects of LFP are locally small categories with (small) limits, and filtered colimits, that have a small set of objects that generate the category under filtered colimits.

Definition 6. The 2-full subcategory of LEX spanned by small categories will be denoted by Lex.

The goal is to prove the following theorem.

Theorem 7 (Gabriel-Ulmer duality)**.** *The adjunction that we defined restricts to an adjoint equivalence:*

$$\mathcal{F} : \mathsf{LFP} \rightleftarrows \mathsf{Lex}^{op} : \mathcal{G}$$

where $\mathcal{F}(\mathcal{A}) = \mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ *and* $\mathcal{G}(\mathcal{C}) = \mathsf{LEX}(\mathcal{C}, \mathsf{Set})$.

As a direct consequence we get:

Proposition 8. *A category is in* LFP *if and only if it is locally finitely presentable.* ■

Lemma 9. *The inclusion* $\mathsf{LEX}(\mathcal{C}, \mathsf{Set}) \hookrightarrow \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ *creates limits and filtered colimits.*

Lemma 10. *The Yoneda embedding* $\mathcal{Y} : \mathcal{C}^{op} \to \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ *is fully faithful and sends colimits to limits.*

Lemma 11. *For any* $M \in \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$ *we can construct the diagram* $D_M : \mathcal{C}^{op}/M \to \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$. *The colimit of this diagram is M which implies that the image of the Yoneda embedding is dense: every functor is canonically a colimit of representable functors.*

Lemma 12. *Let* $\mathcal{C} \in \mathsf{Lex}$ *and* $M \in \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$. *The diagram D_M is filtered if and only if M preserves finite limits, i.e. if M is in* $\mathsf{LEX}(\mathcal{C}, \mathsf{Set})$.

Lemma 13. *If $i : \mathcal{A} \hookrightarrow \mathcal{B}$, $p : \mathcal{B} \to \mathcal{A}$ is a retract then:*

$$\mathcal{B} \rightrightarrows^{\mathsf{Id}}_{ir} \mathcal{B} \to \mathcal{A}$$

is a coequalizer.

Lemma 14. *Given* $M \in \mathsf{CAT}(\mathcal{C}, \mathsf{Set})$, *$M$ finitely presentable if and only if it is a finite colimit of representable functors. In particular, if $\mathcal{C}$ has finite limits and M is in* $\mathsf{LEX}(\mathcal{C}, \mathsf{Set})$, *the functor M is finitely presentable if and only if it is representable.*

Lemma 15. *If $\mathcal{A}$ is locally small, has limits and filtered colimits, and these are preserved by a functor $G : \mathcal{A} \to \mathcal{B}$. Suppose also that $\mathcal{B}$ is locally finitely presentable. Then is $\mathcal{A}$ is generated by a small set of objects under filtered colimits the functor G has a left adjoint $F : \mathcal{B} \to \mathcal{A}$.*

Corollary 16. *If* $\mathcal{A} \in \mathsf{LFP}$, *any functor in* $\mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ *has a left adjoint.*

Lemma 17. *Assume $\mathcal{A} \in \mathsf{LFP}$. If $a \in \mathcal{A}_{f.p.}$ the representable functor $\mathcal{A}(a, -)$ is in $\mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ so the Yoneda embedding factors:*

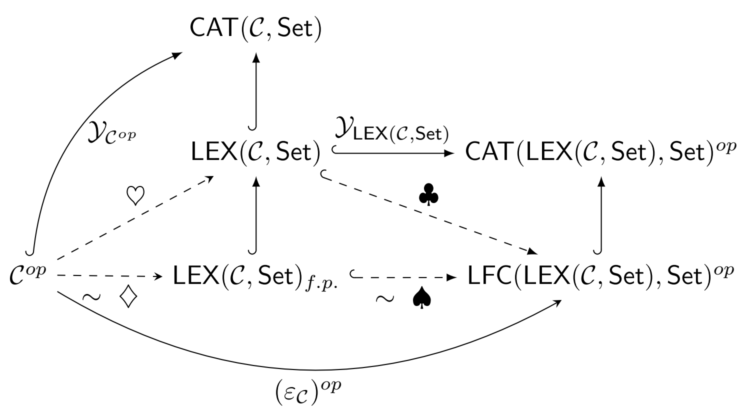
$$\mathcal{A}^{op}_{f.p.} \to \mathsf{LFC}(\mathcal{A}, \mathsf{Set}).$$

This functor is an equivalence.

Proof. Since this functor is essentially the Yoneda embedding we already know that it is fully faithful, so it remains to show that it is essentially surjective. For this take $X : \mathcal{A} \to \mathsf{Set}$ a functor in $\mathsf{LFC}(\mathcal{A}, \mathsf{Set})$. By the above lemma X has a left adjoint $Y : \mathsf{Set} \to \mathcal{A}$ so can consider the natural isomorphisms $\mathcal{A}(Y(1), -) \simeq \mathsf{Set}(1, X(-)) \simeq X(-)$. This implies that X is in the essential image of the Yoneda embedding. Moreover, $Y(1)$ is in $\mathcal{A}_{f.p.}$ since X preserves filtered colimits. ∎

Proposition 18. *$\varepsilon_{\mathcal{C}}$ is an equivalence of categories.*

Proof. Let $\mathcal{A} = \mathsf{LEX}(\mathcal{C}, \mathsf{Set})$. Then $\mathcal{A}$ is locally finitely presentable by Lemma 11 and Lemma 12 (the generating set is the set of representable functors, they clearly respect limits). The Yoneda embedding $\mathcal{C}^{op} \to \mathcal{A}$ factors through $\mathcal{A}_{f.p.} \hookrightarrow \mathcal{A}$ and $\mathcal{C}^{op} \to \mathcal{A}_{f.p.}$ is an equivalence of categories.



Where:

- ♡ is because representable functors preserve limits.
- ♢ is because of Lemma 14.
- ♣ is because representable functors preserve filtered colimits and limits.
- ♠ is because of Lemma 17.

∎

Proposition 19. *If $\mathcal{A}$ is locally finitely presentable then $\mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ is in Lex, i.e. it is essentially small and has finite limits.*

Proof. Consider the inclusion $\mathcal{A}_{f.p.} \hookrightarrow \mathcal{A}$ and the functor $h : \mathcal{A}^{op}_{f.p.} \to \mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ of Lemma 15, which we know is an equivalence. Call S to the small set that generates $\mathcal{A}$. Then that any object in $\mathcal{A}_{f.p.}$ is a finite colimit of objects of S by the same argument used in the proof of Lemma 14, and thus $\mathcal{A}_{f.p.}$ is essentially small.

Since $\mathcal{A}^{op}_{f.p.} \simeq \mathsf{LFC}(\mathcal{A}, \mathsf{Set})$ we deduce that it is essentially small and has all finite limits, so $\mathcal{F}(\mathcal{A}) = \mathsf{LFC}(\mathcal{A}, \mathsf{Set}) \in \mathsf{Lex}$. ■

Proposition 20. *If $\mathcal{A}$ is locally finitely presentable, the inclusion $i : \mathcal{A}_{f.p.} \hookrightarrow \mathcal{A}$ is dense. Concretelly the functor:*

$$\tau : \mathcal{A} \to \mathsf{CAT}(\mathcal{A}^{op}_{f.p.}, \mathsf{Set})$$
$$a \mapsto \mathcal{A}(i(-), a)$$

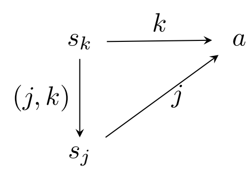
is fully faithful.

Proof. We want to show that:

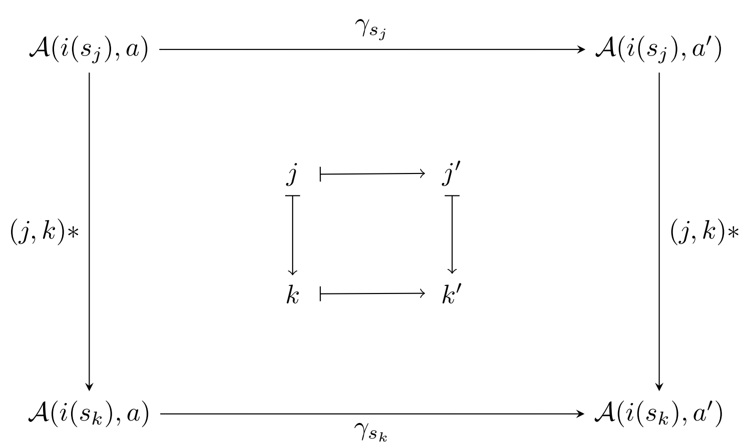
$$\mathcal{A}(a, a') \to \mathrm{Nat}(\mathcal{A}(i(-), a), \mathcal{A}(i(-), a'))$$
$$m \mapsto m_*$$

is a bijection. This is the same as proving that a natural transformation γ in the codomain comes from a unique morphism $m \in \mathcal{A}(a, a')$. The argument is essentially the argument that proves the Yoneda lemma.

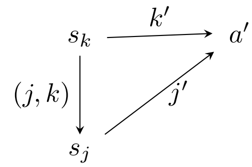
So assume given $\gamma \in \mathrm{Nat}(\mathcal{A}(i(-), a), \mathcal{A}(i(-), a'))$. Write a as a colimit of finitelly presentable objects $a \simeq \mathrm{colim}_j\, s_j$, so part of this diagram looks like:



Consider the naturality square induced by the morphism $s_k \xrightarrow{(j,k)} s_j$:

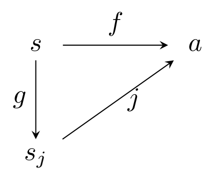


This gives us a cocone:

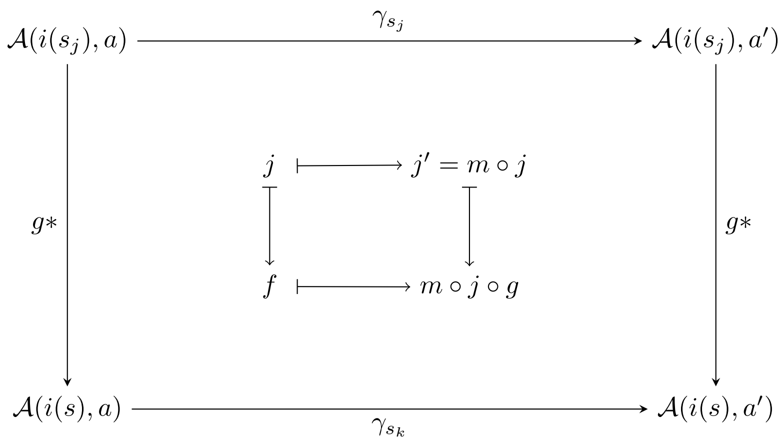


which induces a map $m : a \to a'$. Before showing that m is the morphism that we want let us show that if $m_* = \gamma$ then it is unique: This follows from the uniqueness of m with respect to making the cones commute.

Now we want to prove that $m_* = \gamma$. This means that for every $s \in \mathcal{A}_{f.p.}$, and every $f \in \mathcal{A}(i(s), a)$ we have $\gamma_s(f) = m \circ f$. Start by factoring $f = j \circ g$, for some j, since s is finitelly presentable:



and consider the naturality square induced by $g : s \to s_j$:



Since γ coincides with m_* for the s_j we get that $\gamma_s(f) = m \circ f$, and thus $m_* = \gamma$.

■

Proposition 21. *$\eta_{\mathcal{A}}$ is an equivalence of categories.*

Proof. Consider again h and the induced functor:

$$h^* : \mathsf{LEX}(\mathsf{LFC}(\mathcal{A}, \mathsf{Set}), \mathsf{Set}) \to \mathsf{LEX}(\mathcal{A}^{op}_{f.p.}, \mathsf{Set}).$$

Composing with $\eta_{\mathcal{A}} : \mathcal{A} \to \mathsf{LEX}(\mathsf{LFC}(\mathcal{A}, \mathsf{Set}), \mathsf{Set})$ we get the functor:

$$\tau : \mathcal{A} \to \mathsf{LEX}(\mathcal{A}^{op}_{f.p.}, \mathsf{Set}).$$

Unraveling the definitions we see that τ is exactly $a \mapsto \mathcal{A}(i(-), a)$.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h^*} & \mathsf{LEX}(\mathsf{LFC}(\mathcal{A}, \mathsf{Set}), \mathsf{Set}) \\ & \searrow^{\eta_{\mathcal{A}}} & \downarrow \tau \\ & & \mathsf{LEX}(\mathcal{A}^{op}_{f.p.}, \mathsf{Set}) \end{array}$$

We want to show that $\eta_{\mathcal{A}}$ is an equivalence, so it suffices to show that τ is one. By Proposition 20 τ being fully faithful, so it remains to show that it is essentially surjective. For this consider $M \in \mathsf{LEX}(\mathcal{A}^{op}_{f.p.}, \mathsf{Set})$. By Lemma 12 the functor M is a filtered colimit of representable functors, so we can write:

$$M \simeq \mathrm{colim}_k\, \mathcal{A}^{op}_{f.p.}(a_k, -) \simeq \mathrm{colim}_k\, \mathcal{A}(i(-), a_k) \simeq \mathcal{A}(i(-), \mathrm{colim}_k\, a_k).$$

This implies $M \simeq \tau(\mathrm{colim}_k\, a_k)$. ■